Chapter 4

ARITHMETIC AND GEOMETRIC PROGRESSIONS

A finite sequence such as

$$2, 5, 8, 11, 14, \ldots , 101$$

in which each succeeding term is obtained by adding a fixed number to the preceding term is called an **arithmetic progression**. The general form of an arithmetic progression with n terms is therefore

$$a,\ a + d,\ a + 2d,\ a + 3d,\ \ldots ,\ a + (n - 1)d$$

where a is the first term and d is the fixed **difference** between successive terms.

In the arithmetic progression above, the first term is 2 and the common difference is 3. The second term is $2 + 3\cdot 1$. the third term is $2 + 3\cdot 2$. the fourth is $2 + 3\cdot 3$. and the nth is $2 + 3(n - 1)$. Since $101 - 2 = 99 = 3\cdot 33$ or $101 = 2 + 3(34 - 1)$, one has to add 3 thirty-three times to obtain the nth term. This shows that there are thirty-four terms here. The sum S of these thirty-four terms may be found by the following technique. We write the sum with the terms in the above order and also in reverse order, and add:

$$\begin{array}{rcccccccccc}
S & = & 2 & + & 5 & + & 8 & +\cdots+ & 98 & + & 101 \\
S & = & 101 & + & 98 & + & 95 & +\cdots+ & 5 & + & 2 \\
\hline
2S & = & (2+101) & + & (5+98) & + & (8+95) & +\cdots+ & (98+5) & + & (101+2) \\
 & = & 103 & + & 103 & + & 103 & +\cdots+ & 103 & + & 103 \\
2S & = & 34\cdot 103 & = & 3502. & & & & & &
\end{array}$$

Hence $S = 3502/2 = 1751$.

Using this method, one can show that the sum

$$T_n = 1 + 2 + 3 + 4 + \ldots + n$$

of the first n positive integers is $n(n + 1)/2$. Some values of T_n are given in the table which follows.

n	1	2	3	4	5	6	...
T_n	1	3	6	10	15	21	...

The sequence T_n may be defined for all positive integers n by

$$T_1 = 1,\ T_2 = T_1 + 2,\ T_3 = T_2 + 3,$$

$$T_4 = T_3 + 4,\ \ldots,\ T_{n+1} = T_n + (n + 1),\ \ldots\ .$$

The values 1, 3, 6, 10, 15, ... of T_n are called **triangular numbers** because they give the number of objects in triangular arrays of the type shown in Figure 2.

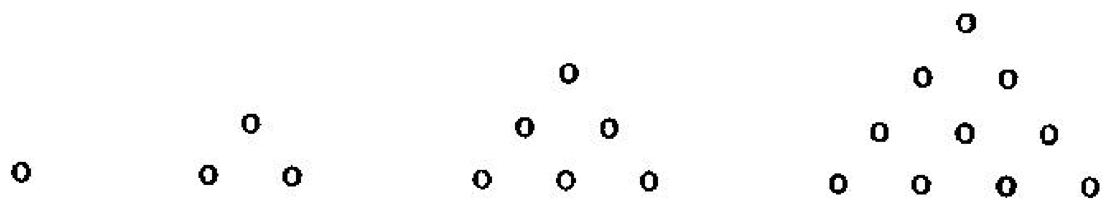

Figure 2

An arithmetic progression may have a negative common difference d. One with $a = 7/3$, $d = -5/3$, and $n = 8$ is:

$$7/3, 2/3, -1, -8/3, -13/3, -6, -23/3, -28/3.$$

The **average** (or **arithmetic mean**) of n numbers is their sum divided by n. For example, the average of 1, 3, and 7 is 11/3. If each of the terms of a sum is replaced by the average of the terms, the sum is not altered. We note that the average of all the terms of an arithmetic progression is the average of the first and last terms, and that the average is the middle term when the number of terms is odd, that is, whenever there is a middle term.

If a is the average of r and s, then it can easily be seen that r, a, s are consecutive terms of an arithmetic progression. (The proof is left to the reader.) This is why the average is also called the arithmetic mean.

A finite sequence such as

$$3, 6, 12, 24, 48, 96, 192, 384$$

in which each term after the first is obtained by multiplying the preceding term by a fixed number, is called a **geometric progression**. The general form of a geometric progression with n terms is therefore

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}.$$

Here a is the first term and r is the fixed multiplier. The number r is called the **ratio** of the progression, since it is the ratio (i.e., quotient) of a term to the preceding term.

We now illustrate a useful technique for summing the terms of a geometric progression.

Example. Sum $5 + 5 \cdot 2^2 + 5 \cdot 2^4 + 5 \cdot 2^6 + 5 \cdot 2^8 + \ldots + 5 \cdot 2^{100}$.

Solution: Here the ratio r is $2^2 = 4$. We let S designate the desired sum and write S and rS as follows:

$$\begin{aligned} S &= 5 + 5 \cdot 2^2 + 5 \cdot 2^4 + 5 \cdot 2^6 + \ldots + 5 \cdot 2^{100} \\ 4S &= \quad\;\; 5 \cdot 2^2 + 5 \cdot 2^4 + 5 \cdot 2^6 + \ldots + 5 \cdot 2^{100} + 5 \cdot 2^{102}. \end{aligned}$$

Subtracting, we note that all but two terms on the right cancel out and we obtain

$$3S = 5 \cdot 2^{102} - 5$$

or

$$3S = 5(2^{102} - 1).$$

Hence we have the compact expression for the sum:

$$S = \frac{5(2^{102} - 1)}{3}.$$

If the ratio r is negative, the terms of the geometric progression alternate in signs. Such a progression with $a = 125$, $r = -1/5$, and $n = 8$ is

$$125, -25, 5, -1, 1/5, -1/25, 1/125, -1/625.$$

The geometric mean of two positive real numbers a and b is $\sqrt{ab}$, the positive square root of their product; the geometric mean of three positive numbers a, b, and c is $\sqrt[3]{abc}$. In general, the **geometric mean** of n positive numbers is the nth root of their product. For example, the geometric mean of 2, 3, and 4 is $\sqrt[3]{2 \cdot 3 \cdot 4} = \sqrt[3]{8 \cdot 3} = 2\sqrt[3]{3}$.

Problems for Chapter 4

1. (a) Find the second, third, and fourth terms of the arithmetic progression with the first term -11 and difference 7.
   (b) Find the next three terms of the arithmetic progression -3, -7, -11, -15, ...

2. (a) Find the second, third, and fourth terms of the arithmetic progression with the first term 8 and difference -3.
   (b) Find the next three terms of the arithmetic progression 7/4, 1, 1/4, -1/2, -5/4, -2, ...

3. Find the 90th term of each of the following arithmetic progressions:

   (a) 11, 22, 33, 44, ...
   (b) 14, 25, 36, 47, ...
   (c) 9, 20, 31, 42, ...

4. For each of the following geometric progressions, find e so that 3^e is the 80th term.

   (a) 3, 9, 27, ...
   (b) 1, 3, 9, ...
   (c) 81, 243, 729, ...

5. Find x, given that 15, x, 18 are consecutive terms of an arithmetic progression.

6. Find x and y so that 14, x, y, 9 are consecutive terms of an arithmetic progression.

7. Sum the following:

   (a) 7/3 + 2/3 + (-1) + (-8/3) + (-13/3) + ... + (-1003/3).
   (b) (-6) + (-2) + 2 + 6 + 10 + ... + 2002.
   (c) The first ninety terms of 7/4, 1, 1/4, -1/2, -5/4, -2, ...
   (d) The first n odd positive integers, that is, $1 + 3 + 5 + ... + (2n - 1)$.

8. Sum the following:

   (a) 12 + 5 + (-2) + (-9) + ... + (-1073).
   (b) (-9/5) + (-1) + (-1/5) + 3/5 + 7/5 + 11/5 + ... + 2407.
   (c) The first eighty terms of -3, -7, -11, -15, ...
   (d) The first n terms of the arithmetic progression a, $a + d$, $a + 2d$, ...

9. Find the fourth, seventh, and ninth terms of the geometric progression with first term 2 and ratio 3.

10. Find the fourth and sixth terms of the geometric progression with first term 2 and ratio -3.

11. Find the next three terms of the geometric progression 2, 14, 98, ...

12. Find the next three terms of the geometric progression 6, -2, 2/3, -2/9, ...

13. Find both possible values of x if 7, x, 252 are three consecutive terms of a geometric progression.

14. Find all possible values of y if 400, y, 16 are three consecutive terms of a geometric progression.

15. Find a compact expression for each of the following:

(a) $1 + 7 + 7^2 + 7^3 + ... + 7^{999}$.
(b) $1 - 7 + 7^2 - 7^3 + ... - 7^{999}$.
(c) $1 + 7 + 7^2 + 7^3 + ... + 7^{n-1}$.

16. Find a compact expression for each of the following:

(a) $1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + ... + \frac{1}{3^{88}}$.

(b) $8 + \frac{8}{3^2} + \frac{8}{3^4} + \frac{8}{3^6} + ... + \frac{8}{3^{188}}$.

(c) $8 + 8 \cdot 3^{-2} + 8 \cdot 3^{-4} + 8 \cdot 3^{-6} + ... + 8 \cdot 3^{-2m}$.

17. Find n, given that $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \binom{n}{2}$.

*18. Find m, given that $1 + 2 + 3 + 4 + ... + 1000 = \binom{m}{m-2}$.

19. Given that a is the average of the numbers r and s, show that r, a, s are three consecutive terms of an arithmetic progression and that their sum is $3a$.

20. Show that r^3, r^2s, rs^2, s^3 are four consecutive terms of a geometric progression and that their sum is $(r^4 - s^4)/(r - s)$.

21. Find the geometric mean of each of the following sets of positive numbers:

(a) 6, 18.
(b) 2, 6, 18, 54.
(c) 2, 4, 8.
(d) 1, 2, 4, 8, 16.

22. Find the geometric mean of each of the following sets of numbers:

(a) 3, 4, 5.
(b) 3, 4, 5, 6.
(c) $1, 7, 7^2, 7^3$.
(d) a, ar, ar^2, ar^3, ar^4.

23. Find the geometric mean of 8, 27, and 125.

24. Find the geometric mean of a^4, b^4, c^4, and d^4.

25. Let b be the middle term of a geometric progression with $2m + 1$ positive terms and let r be the common ratio. Show that:

(a) The terms are $br^{-m}, br^{-m+1}, \ldots , br^{-1}, b, br, \ldots , br^m$.
(b) The geometric mean of the $2m + 1$ numbers is the middle term.

26. Show that the geometric mean of the terms in a geometric progression of positive numbers is equal to the geometric mean of any two terms equally spaced from the two ends of the progression.

27. Find a compact expression for the sum $x^n + x^{n-1}y + x^{n-2}y^2 + \ldots + xy^{n-1} + y^n$.

28. Find a compact expression for the arithmetic mean of $x^n, x^{n-1}y, x^{n-2}y^2, \ldots, xy^{n-1}, y^n$.

29. A 60-mile trip was made at 30 miles per hour and the return at 20 miles per hour.

(a) How many hours did it take to travel the 120 mile round trip?
(b) What was the average speed for the round trip?

30. Find x, given that $1/30$, $1/x$, and $1/20$ are in arithmetic progression. What is the relation between x and the answer to Part (b) of problem 29?

31. Verify the factorization $1 - x^7 = (1 - x)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)$ and use it with $x = 1/2$ to find a compact expression for

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6.$$

32. Use the factorization $1 + x^{99} = (1 + x)(1 - x + x^2 - x^3 + x^4 - \ldots + x^{98})$ to find compact expressions for the following sums:

(a) $1 - 5^{-1} + 5^{-2} - 5^{-3} + \ldots - 5^{-97} + 5^{-98}$.
(b) $a - ar + ar^2 - ar^3 + \ldots - ar^{97} + ar^{98}$.

33. Let $a_1, a_2, a_3, \ldots, a_{3m}$ be an arithmetic progression, and for $n = 1, 2, \ldots, 3m$ let A_n be the arithmetic mean of its first n terms. Show that A_{2m} is the arithmetic mean of the two numbers A_m and A_{3m}.

34. Let $g_1, g_2, \ldots, g_{3m}$ be a geometric progression of positive terms. Let A, B, and C be the geometric means of the first m terms, the first $2m$ terms, and all $3m$ terms, respectively. Show that $B^2 = AC$.

35. Let S be the set consisting of those of the integers 0, 1, 2, ..., 30 which are divisible exactly by 3 or 5 (or both), and let T consist of those divisible by neither 3 nor 5.

(a) Write out the sequence of numbers in S in their natural order.
(b) In the sequence of Part (a), what is the arithmetic mean of terms equally spaced from the two ends of the sequence?
(c) What is the arithmetic mean of all the numbers in T?
(d) Find the sum of the numbers in T.

36. Find the sum 4 + 5 + 6 + 8 + 10 + 12 + 15 + ... + 60,000 of all the positive integers not exceeding 60,000 which are integral multiples of at least one of 4, 5, and 6.

37. Let $u_1, u_2, ..., u_t$ satisfy $u_{n+2} = 2u_{n+1} - u_n$ for $n =1, 2, ..., t - 2$. Show that the t terms are in arithmetic progression.

38. Find a compact expression for the sum $v_1 + v_2 + ... + v_t$ in terms of v_1 and v_2, given that $v_{n+2} = (v_{n+1})^2/v_n$ for $n = 1, 2, ... ,t - 2$.

39. Let $a_n = 2^n$ be the nth term of the geometric progression $2, 2^2, 2^3, ... , 2^t$. Show that $a_{n+2} - 5a_{n+1} + 6a_n = 0$ for $n = 1, 2, ... , t - 2$.

40. For what values of r does the sequence $b_n = r^n$ satisfy $b_{n+2} - 5b_{n+1} + 6b_n = 0$ for all n?

41. Let a be one of the roots of $x^2 - x - 1 = 0$. Let the sequence $c_0, c_1, c_2, ...$ be the geometric progression $1, a, a^2, ...$. Show that:

(a) $c_{n+2} = c_{n+1} + c_n$.
(b) $c_2 = a^2 = a + 1$.
(c) $c_3 = a^3 = 2a + 1$.
(d) $c_4 = 3a + 2$.
(e) $c_5 = 5a + 3$.
(f) $c_6 = 8a + 5$.

42. For the sequence $c_0, c_1, ...$ of the previous problem, express c_{12} in the form $aF_u + F_v$, where F_u and F_v are Fibonacci numbers, and conjecture a similar expression for c_m.

*43. In the sequence 1/5, 3/5, 4/5, 9/10, 19/20, 39/40, ... each succeeding term is the average of the previous term and 1. Thus:

$$\frac{3}{5} = \frac{1}{2}\left(\frac{1}{5} + 1\right),\ \frac{4}{5} = \frac{1}{2}\left(\frac{3}{5} + 1\right),\ \frac{9}{10} = \frac{1}{2}\left(\frac{4}{5} + 1\right),\ \ldots .$$

(a) Show that the twenty-first term is $1 - \dfrac{1}{5{\cdot}2^{18}}$.

(b) Express the *n*th term similarly.

(c) Sum the first five hundred terms.

*44. In the sequence 1, 2, 3, 6, 7, 14, 15, 30, 31, ... a term in an even numbered position is double the previous term, and a term in an odd numbered position (after the first term) is one more than the previous term.

(a) What is the millionth term of this sequence?

(b) Express the sum of the first million terms compactly.